



Supplement of

Three-dimensional density field reconstruction of vehicle exhaust plumes using 3D gas schlieren imaging sensor system for remote emission sensing applications

Hafiz Hashim Imtiaz et al.

Correspondence to: Hafiz Hashim Imtiaz (hafiz.imtiaz@tugraz.at)

The copyright of individual parts of the supplement might differ from the article licence.

Supplementary Material

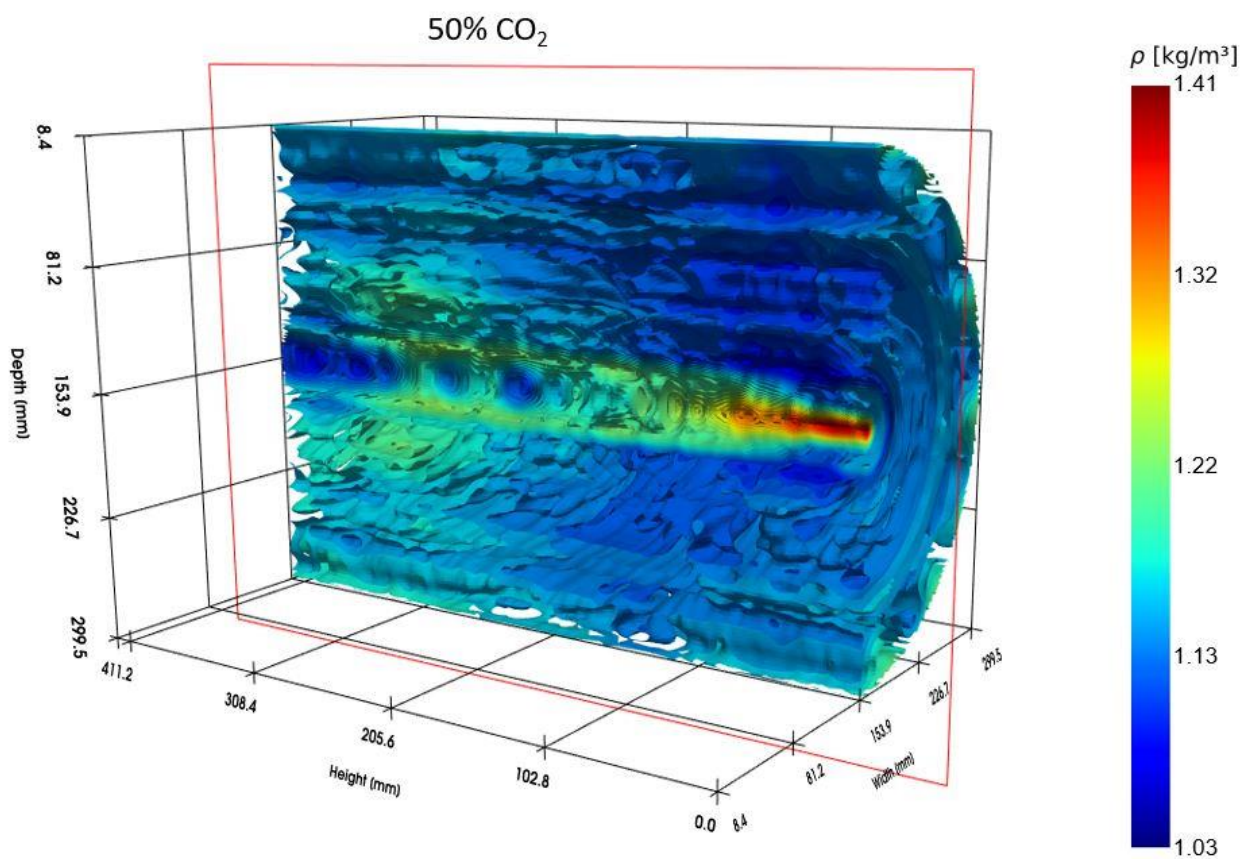


Figure S1. Cross-sectional density distribution resulting from a 50% carbon dioxide flow

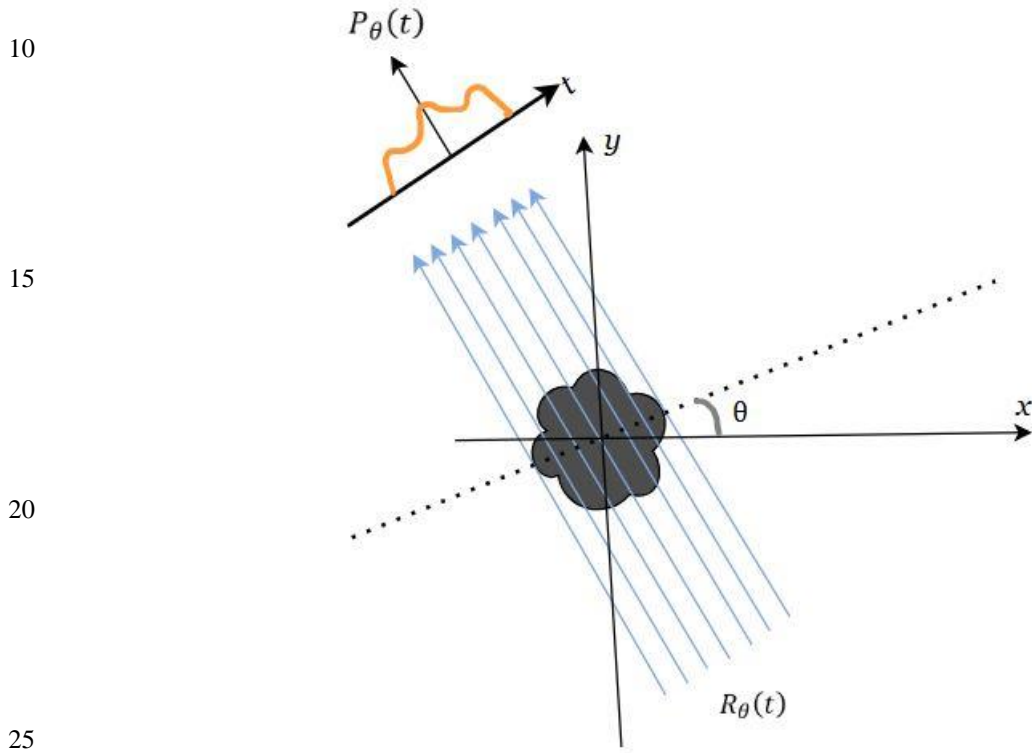


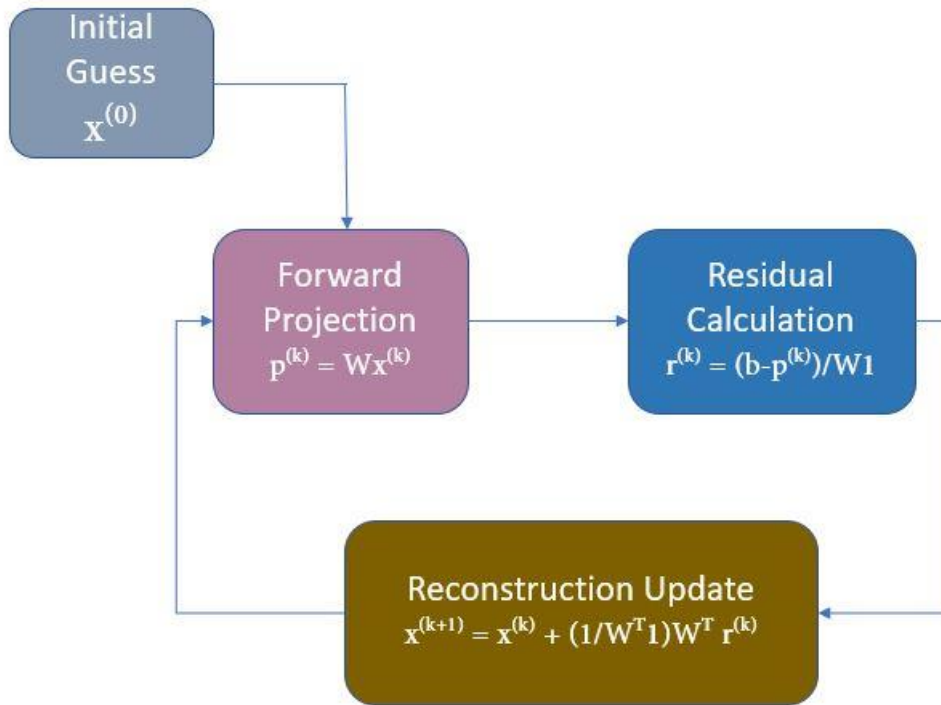
Figure S2. Illustration of the forward projection principle using the Radon transform. A set of parallel rays pass through the object at angle θ , integrating the attenuation along each line to form the projection $P_\theta(t)$

S1. Simultaneous Algebraic Reconstruction Technique (SART)

30 The Simultaneous Algebraic Reconstruction Technique (SART) is well-suited for scenarios involving a limited number of cameras, where projection data is inherently sparse. SART reconstructs cross-sectional images from projection measurements by formulating the problem as a system of linear equations. In the discrete formulation, the unknown image, $f(x, y)$ is represented as a vector $\mathbf{x} \in \mathcal{R}^N$ where N is the number of unknowns. The measured projection data from M rays is denoted by $\mathbf{b} \in \mathcal{R}^M$, and the weight matrix $\mathbf{W} \in \mathcal{R}^{M \times N}$ defines the contribution of each element in \mathbf{x} to each projection in \mathbf{b} . The forward
 35 model is thus given by Eq. (S1).

$$\mathbf{b} = \mathbf{W} \mathbf{x} \text{ (S1)}$$

The weights in \mathbf{W} can be computed using criteria such as the length of intersection of each ray with the discretized elements or the area influenced by the ray path. While in theory this system could be solved directly via matrix inversion, this is generally infeasible in practice due to the large dimensions of \mathbf{W} and its frequent non-invertibility. To address this, SART employs an iterative reconstruction algorithm based on the Kaczmarz method, which incrementally updates the solution by reducing the residual error at each iteration [S1]. The procedure for SART can be seen in Fig. S3. The process begins with an initial guess for \mathbf{x} , typically a zero vector. This estimate is forward projected to yield $\mathbf{p} = \mathbf{W} \mathbf{x}$, simulating the projection data. The residual between the measured data \mathbf{b} and the simulated projection \mathbf{p} is then computed and normalized by the row sums of \mathbf{W} , given by $\mathbf{W}\mathbf{1}$, where $\mathbf{1}^T = [1 \ 1 \ \dots \ 1]_{1 \times N}$. This normalized residual is then back-projected using the transpose of the weight matrix and applied as a correction to the current estimate of \mathbf{x} , completing one iteration. Repeating this process over multiple iterations yields progressively refined reconstructions with reduced error, making SART an effective solution for tomographic imaging under sparse data conditions.

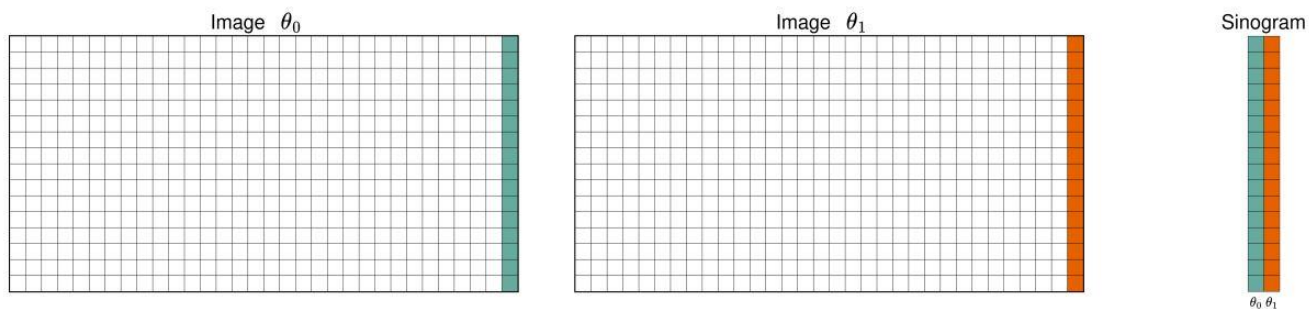


50

Figure S3. Procedure of the Simultaneous Algebraic Reconstruction Technique (SART) Algorithm

S2. Implementation of SART Algorithm

Reconstructing the density distribution involves utilizing the line integrals generated from aligned data. To carry out the reconstruction via the SART (Simultaneous Algebraic Reconstruction Technique) algorithm, sinograms must be created for each cross-section of the volume being reconstructed. In the laboratory setup employed, each camera captures multiple cross-sections along the width of the image. Consequently, each column of pixels corresponds to a distinct cross-section of the reconstruction volume, viewed from a specific angle. When the data is correctly aligned, a sinogram for a given cross-section is formed by merging the corresponding pixel columns from each camera as illustrated in Fig. S4.



65 **Figure S4. Sinogram representation of a cross-sectional scan. Each column of pixels in the image corresponds to a projection at a specific angle and collectively forms part of the sinogram representing the associated cross-section**

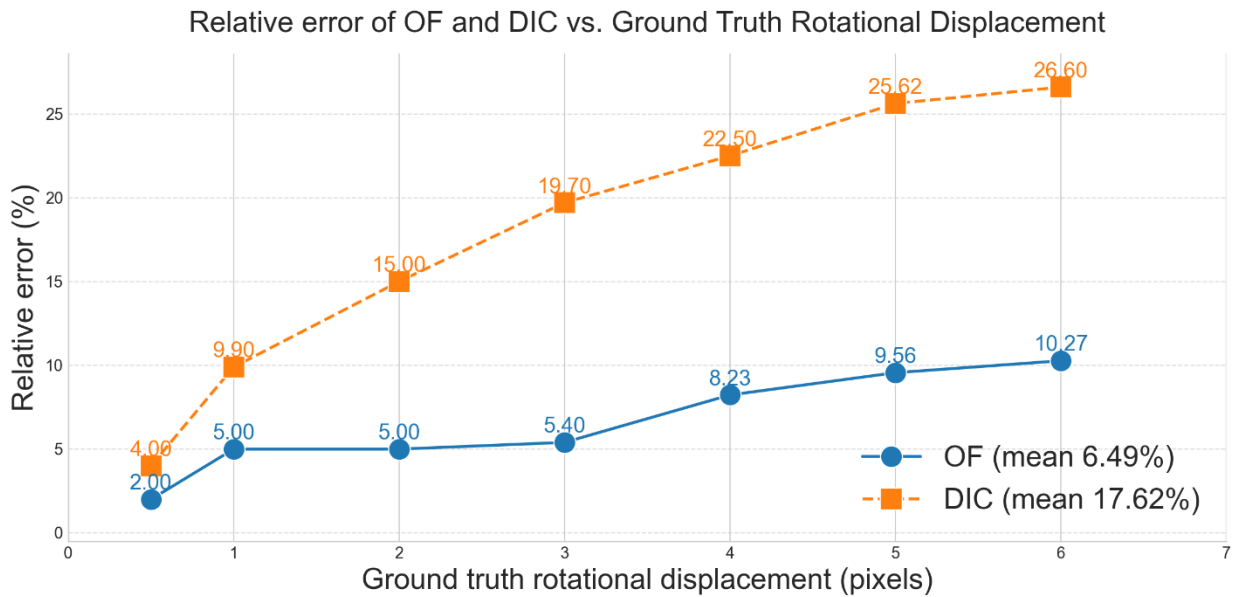


Figure S5. Relative error of displacement (rotational) estimated using optical flow (OF) and digital image correlation (DIC) with respect to the ground truth

70

References (Supplement)

[S1] Andersen, A. H. and Kak, A. C.: Simultaneous algebraic reconstruction technique (SART): A superior implementation of the ART algorithm, *Ultrason. Imaging*, 6, 81–94, <https://doi.org/10.1177/016173469400600108>, 1984.